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Towards a structure theory for projective varieties of degree = codimension + 2

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Abstract

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The problem under consideration in this paper is that of finding a structure theory for varieties X of $\mathbb{P}_k^n$ (k is an algebraically closed field of arbitrary characteristic) with degree(X) = codimension(X) + 2. Takao Fujita has a satisfactory classification theory for projective varieties of Δ-genus zero and one. In either case the singularities of X turn out to be of very special type. Our approach also sheds some light on the structure of these singularities.

1. Introduction and main results

The problem under consideration in this paper is that of finding a structure theory for varieties X of $\mathbb{P}_k^n$ (k is an algebraically closed field of arbitrary characteristic) with degree(X) = codimension(X) + 2. The structure of X is quite well understood in assuming degree(X) = codimension(X) + 1 (see, e.g., [3]). Fujita has a satisfactory classification theory for projective varieties of Δ-genus zero and one (see, e.g., [6, 7]). We recall that the Δ-genus of the polarized varieties $(X, \mathcal{O}_X(1))$ is defined by $\Delta :=$ degree(X) − codim(X) − 1 ≥ 0 in assuming that the restriction mapping $H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(X, \mathcal{O}_X(1))$ is bijective. In particular, Fujita shows that X is either a normal *del Pezzo* variety or the image of a variety of Δ-genus zero via a projection. In either case the singularity of X turns out to be of very special type. Our approach sheds some light on the structure of these singularities (see our Theorem B below).

On the other hand, Sally [16] has studied Cohen–Macaulay local rings of embedding dimension = multiplicity + dimension − 2. For example, the first theorem of [16] generalizes a statement of [15], where the local ring was assumed to be Gorenstein. However, it is well known that such graded k-algebra domains A are Cohen–Macaulay if and only if they are Gorenstein (see again our Theorem B below).

In some sense the heart of our paper are the following three statements.

Theorem A. *Let k be an algebraically closed field of arbitrary characteristic. Let X be a reduced, pure-dimensional and nondegenerate subscheme of $\mathbb{P}^n_k$. We assume that X is connected in codimension one if* $\dim X \geq 1$. *We set* $A := k[x_0, x_1, \ldots, x_n]/I(X)$, *where $I(X)$ is the defining ideal of X,* $d := \dim A$ *and* $r := \operatorname{depth} A$. *The following conditions are equivalent*:

(i) degree $X = 2 + \operatorname{codim} X$,

(ii) reg $A = 2$, $H^i_P(A) = 0$ *for all* $i \neq r, d$, *and A is given by one of the following cases*:

(a) *A is Cohen–Macaulay (i.e. X is arithmetically Cohen–Macaulay) and* $[H^d_P(A)]_{2-d} \cong k$.

(b) $d \geq 2$, $r = 1$, $H^1_P(A) \cong k(-1)$ *and* $[H^d_P(A)]_{2-d} = 0$.

(c) $d \geq 3$, $r = d - 1$, $[H^d_P(A)]_{2-d} = 0$, *and*

$$\operatorname{rank}_k[H^{d-1}_P(A)]_p = \begin{cases} 1 & \textit{for } p = 3 - d\,, \\ d-2 & \textit{for } p = 2 - d\,. \end{cases}$$

(d) $d \geq 4$, $2 \leq r \leq d - 2$, $[H^d_P(A)]_{2-d} = 0$, *and*

$$\operatorname{rank}_k[H^r_P(A)]_p = \binom{-p}{r-2} \quad \textit{for } p \leq 2 - r\,,$$

where we set $\binom{p}{0} := 1$.

The second statement provides the structure of the local cohomology and in certain cases also the structure of the singularities.

Theorem B. *Let k be an algebraically closed field of arbitrary characteristic. Let $X \subseteq \mathbb{P}^n_k$ be a reduced, pure-dimensional and nondegenerate subscheme being connected in codimension* 1. *Let A be the graded k-algebra* $k[x_0, \ldots, x_n]/I(X)$, *where $I(X)$ is the defining ideal of X. We set* $d := \dim A \geq 2$ *and* $r := \operatorname{depth} A$. *Assume* $\operatorname{degree}(X) = \operatorname{codim}(X) + 2$, *then we have*:

(i) *$H^i_P(A) = 0$ for all $i \neq r, d$. Moreover, if $r \leq d-1$, then we get*

$$H^r_P(A) \cong k[y_0, \ldots, y_{r-2}]^{\vee}(r-2)\,,$$

where $y_0, \ldots, y_{r-2}$ are algebraically independent elements of degree 1 of A.

(ii) depth $A = 1$ *if and only if X is arithmetically Buchsbaum and not Cohen–Macaulay.*

(iii) *If $d \geq 3$, then* depth $A = 2$ *if and only if X is locally Buchsbaum and not locally Cohen–Macaulay.*

(iv) *Assume that X is irreducible. Then X is arithmetically Cohen–Macaulay if and only if X is arithmetically Gorenstein.*

The third statement describes the structure of the defining graded k-algebra of $(n+2)$ points of $\mathbb{P}^n_k$. We get in particular the minimal free resolution of such an algebra. Hence our next theorem is also a continuation of the work to study the problem of determining the (graded) minimal free resolution of the ideal of a finite number of points in $\mathbb{P}^n$ (see, e.g., [13]). Moreover, the case of $(n+2)$ points was also studied in [9] and [8]. The statements (ii) and (iii) of the following Theorem C yield some progress on the questions raised in these papers (see, for example, [9, Theorem 7] or [8, Proposition 4.8]).

Theorem C. *Let k be an algebraically closed field of arbitrary characteristic. Let $X \subset \mathbb{P}^n_k$ $(n \geq 2)$ be a set of $(n+2)$ points, spanning P^n. Let A be the graded k-algebra $R/I(X)$, where $I(X)$ is the defining ideal of X and $R := k[x_0, \ldots, x_n]$. Then we have:*

$$\text{(i)} \qquad \operatorname{rank}_k[H^1_P(A)]_p = \begin{cases} 0 & \text{for } p \geq 2\,, \\ 1 & \text{for } p = 1\,, \\ n+1 & \text{for } p = 0\,, \\ n+2 & \text{for } p \leq -1\,. \end{cases}$$

(ii) *The $(n+2)$ points of X are in general position in $\mathbb{P}^n_k$ if and only if A is a Gorenstein algebra.*

(iii) *Let q be the smallest integer such that there are $(q+2)$ points of X, spanning $\mathbb{P}^q$. (We note that $1 \leq q \leq n$, and $q = n$ if and only if the points of X are in general position.) Then we get the following minimal free resolution of A:*

$$\begin{aligned} 0 \quad &\to R^{\beta_n}(-n-1) \oplus R^{\beta'_n}(-n-2) \to R^{\beta_{n-1}}(-n) \oplus R^{\beta'_{n-1}}(-n-1) \to \cdots \\ &\to R^{\beta_q}(-q-1) \oplus R^{\beta'_q}(-q-2) \to R^{\beta_{q-1}}(-q) \to \cdots \\ &\to R^{\beta_1}(-2) \to R \to A \to 0\,, \end{aligned}$$

where the Betti numbers are given for $i = 1, \ldots, n$ by

$$\beta_i = (n+2)\binom{n}{i} - \binom{n+2}{i+1} + \binom{n-q}{i-q-1} \quad \text{and} \quad \beta'_i = \binom{n-q}{i-q}\,.$$

(Note that $\binom{x}{-l} := 0$.)

The proofs of these theorems show that our statements are special cases of more general but technical results, see, e.g., Lemmas 6 and 7. In proving our main results the key is the connectedness theorem of Grothendieck [10], and a careful study of Castelnuovo's regularity and of the Hilbert polynomial of subschemes with degree = codimension + 2 (see our Theorem 3 in Section 4). Our research on Castelnuovo's regularity improves main results of [21]. Moreover, Grothendieck's connectedness theorem also enables us to give a straightforward proof of a well-known (see, e.g., [2]) characterization of varieties of minimal degree (see Theorem 5 in Section 4).

In Section 7 we conclude by studying some examples and open questions. For instance, Example 1 demonstrates that Theorem B(iv) does not remain true when X is not irreducible.

2. Notations and preliminary results

Before proving the theorems we will recall some notations and basic facts on connectedness, Castelnuovo's regularity and graded local cohomology. We work over an algebraically closed field k of arbitrary characteristic.

Let $A = \bigoplus_{n\geq 0} A_n$ be a graded k-algebra, i.e., $A_0 = k$ and A is generated as a k-algebra by A_1. Hence we always set $A := k[x_0, \ldots, x_n]/I$ where $n+1 = \operatorname{rank}_k A_1$ and I is a homogeneous ideal of the polynomial ring $R := k[x_0, \ldots, x_n]$ in $n+1$ indeterminates of degree one.

Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded A-module. The ith local cohomology module of M with support in the irrelevant ideal $P = \bigoplus_{n>0} A_n$, denoted by $H^i_P(M)$, is also a graded A-module. Concerning local cohomology theory of graded modules see, for example, [20].

For an arbitrary graded A-module M we set

$$e(M) := \sup\{n \in \mathbb{Z} \colon [M]_n \neq 0\}\ ,$$

where $[M]_i$ denotes the ith graded part of M, i.e., $[M]_i = M_i$. Let p be an integer. Then let $M(p)$ denote the graded A-module whose underlying module is the same as that of M and whose grading is given by $[M(p)]_i = M_{p+i}$ for all integers i. Let $M^\vee$ be the dual of M (see, e.g., [20, Chapter 0]). Let M be a finitely generated A-module and let m be an integer. We say that M is *m-regular* if $[H^i_P(M)]_j = 0$ for every i, j such that $i + j > m$. We define Castelnuovo's regularity $\operatorname{reg} M$ of M by

$$\begin{aligned}\operatorname{reg}(M) &= \inf\{m \in \mathbb{Z} \colon M \text{ is } m\text{-regular}\}\\ &= \max\{i + e(H^i_P(M)\colon 0 \leq i \leq \dim M\}\ .\end{aligned}$$

Let A be a graded k-algebra. Let d be the Krull-dimension of A. We set $d := \dim A$. Moreover, we define the codimension of A as follows: $\operatorname{codim} A := \operatorname{rank}_k[A]_1 - d \geq 0$. If $d = 0$, then $\operatorname{codim} A = \operatorname{rank}_k[A]_1 = n + 1$.

Let $H(t, A) := \operatorname{rank}_k[A]_t$ be the Hilbert function of A. We write for the Hilbert polynomial $h(t, A)$ of A ($h(t, A) = H(t, A)$ for large t),

$$h(t, A) := h_0(A)\binom{t}{d-1} + h_1(A)\binom{t}{d-2} + \cdots + h_{d-1}(A)\,,$$

where $d = \dim A \geq 1$. The integer $h_0(A)$ is said to be the degree of A. We have $h_0(A) \geq \operatorname{codim} A + 1$. The k-algebra is called *of minimal degree* if $h_0(A) = \operatorname{codim} A + 1$ (see [3] for a centennial account of such varieties).

We will use also Serre cohomology, that is, we define

$$H^0(A) := \varinjlim_n \operatorname{Hom}_A(P^n, A)\,,$$

and for $i > 0$

$$H^i(A) := \varinjlim_n \operatorname{Ext}^i_A(P^n, A)\,.$$

We have a well-known interplay between local cohomology and Serre cohomology (see, e.g., [20, Chapter 0]). For instance, we have isomorphisms

$$H^i(A) \cong H_P^{i+1}(A) \quad \text{for all } i \geq 1\,,$$

and we have the following exact sequence:

$$0 \to H_P^0(A) \to A \to H^0(A) \to H_P^1(A) \to 0\,.$$

Moreover, local duality yields isomorphisms for all $i \geq 0$:

$$H_P^i(A) \cong \operatorname{Ext}_R^r(A, R)^{\vee}(r)$$

where $r := \operatorname{rank}_k[A]_1$ (see, e.g., [20, Corollary 0.3.5]). Furthermore, we obtain from [18, Chapter 3, §6]

$$h(t, A) = \sum_{i \geq 0} (-1)^i \operatorname{rank}_k[H^i(A)]_t\,.$$

The following lemma collects some results that we need. We prove it for the sake of completeness.

Lemma 1. *Let A be a graded k-algebra of dimension $d \geq 2$. Then we have*:

(i) *A has no associated prime ideals $\mathfrak{p}$ with* $\dim A/\mathfrak{p} = 1$ *if and only if $H_P^1(A)$ is noetherian (and hence $H_P^1(A)$ has finite length over A).*

(ii) *Assume that A is pure-dimensional and reduced. Then we have*:

(a) $[H_P^1(A)]_p = 0$ *for all $p < 0$.*

(b) $1+\operatorname{rank}_k[H^1_P(A)]_0$ *is the number of the connected components of* Proj A *where* Proj A *denotes the projective spectrum of* A.

(c) Proj A *is connected if and only if* $[H^0(A)]_0 \cong k$, *and in this case we have* $[H^1_P(A)]_p = 0$ *for all* $p \le 0$ (see also [2, Lemma 4.4]).

Proof. We get (i), for example, by analyzing the proof of Corollary 0.4.15 of [20].

(ii)(a) Consider the exact sequence

$$0 \to A \to H^0(A) \to H^1_P(A) \to 0 .$$

Applying (i) of this lemma we get $[H^1_P(A)]_q = 0$ for $q \ll 0$ since A is pure-dimensional. Hence $[H^0(A)]_q = 0$ for $q \ll 0$. Consider an element $h \in [H^0(A)]_p$ with $p < 0$. Then we have $h^q = 0$ for $q \gg 0$. We note that $H^0(A)$ is also a graded algebra which is reduced since A is reduced. Therefore, we get $h = 0$. The above exact sequence shows that $[H^1_P(A)]_p = 0$ for all $p < 0$.

(b) We set $X = \operatorname{Proj} A$. The assertion (b) follows from the fact that $\operatorname{rank}_k[H^0(A)]_0 = \operatorname{rank}_k H^0(X, \mathcal{O}_X)$ is the number of the connected components of X which follows from [18, p. 272, proof of Proposition 4] and by using an easy induction on the number of connected components of X.

(c) This follows from (a) and (b) of (ii). This completes the proof of Lemma 1. □

This lemma used some connectedness properties of Proj A. Our proofs need also the concept of a scheme being connected in codimension k for some integer $k \ge 1$. We therefore recall some basic facts. Let X be a subscheme of $\mathbb{P}^n_k$. Let $I(X)$ be the defining ideal of X in the polynomial ring $R := k[x_0, \ldots, x_n]$. We set $A := R/I(X)$, $\dim X = \dim A - 1$, and $\operatorname{codim} X = \operatorname{codim} A$ if X is nondegenerate. Let $k \ge 0$ be an integer. We say that X is *connected in codimension* k if $X - Y$ is connected for every closed subset Y of X with $\operatorname{codim}(Y, X) > k$. For example, a scheme consisting of two planes which meet in a point is connected in codimension 2 but not connected in codimension 1. While it is certainly not true that an arbitrary linear section of an irreducible variety remains irreducible, one does have a connectedness part of Bertini's theorem. Therefore our proofs use the projective version of the connectedness theorem of Grothendieck [10, Exp. 13, Theorem 2.1 and Corollary 2.3].

Moreover, we need some basic results on Gorenstein and Buchsbaum rings. The readers may consult the standard books [14] and [20] for the general reference on Gorenstein rings and Buchsbaum rings, respectively.

Finally, we use the following notation: Let $\mathfrak{a}$ be an ideal of A and let N be a submodule of an A-module M. We set

$$N :_M \langle \mathfrak{a} \rangle := \{m \in N\colon \text{there exists an integer } n \ge 0 \text{ with } \mathfrak{a}^n \cdot m \subseteq N\} ,$$

and

$$N :_M \mathfrak{a} := \{m \in M\colon \mathfrak{a} \cdot m \subseteq N\} .$$

3. Proof of Theorem C

We need the following lemma which is a simple observation.

Lemma 2. *Let A be a graded k-algebra. Let l be a form of degree 1 of $R := k[x_0, \ldots, x_n]$, and we set $S := R/l \cdot R$. Then we have*:

(i) *Suppose that l is a nonzerodivisor of A. If $F_. \to A \to 0$ is a minimal free resolution of A as a graded R-module, then*

$$F_./l \cdot F_. \to A/l \cdot A \to 0$$

is a minimal free resolution of A/lA of graded S-modules. Moreover, we get

$$\operatorname{Tor}_i^R(k, A) \cong \operatorname{Tor}_i^S(k, A/lA) \quad \text{for } i \geq 0 .$$

(ii) *Suppose $l \cdot A = 0$. Then we have for $i \geq 0$*

$$\operatorname{Tor}_i^R(k, A) \cong \operatorname{Tor}_i^S(k, A) \oplus (\operatorname{Tor}_{i-1}^S(k, A)(-1)) .$$

Proof. (i) Since l is a nonzerodivisor of R and A we have the following exact sequences of complexes:

$$\begin{array}{ccc}
0 & 0 & \\
\downarrow & \downarrow & \\
F_.(-1) \to & A(-1) \to & 0 \\
\downarrow l & \downarrow l & \\
F_. \to & A \to & 0 \\
\downarrow & \downarrow & \\
F_./lF_. \to & A/lA \to & 0 \\
\downarrow & \downarrow & \\
0 & 0 &
\end{array}$$

Since $F_. \to A \to 0$ is exact we therefore obtain that $F_./lF_. \to A/lA \to 0$ is exact. Moreover, $F_./lF_. \to A/lA \to 0$ is a minimal free resolution since $F_. \to A \to 0$ has this property. Hence we get for all $i \geq 0$

$$\operatorname{Tor}_i^R(k, A) \cong k \otimes F_i \cong k \otimes (F_i/lF_i) \cong \operatorname{Tor}_i^S(k, A/lA) .$$

(ii) Let $l_1, \ldots, l_n$ be elements of $[R]_1$ such that $\{l, l_1, \ldots, l_n\}$ is a basis of $[R]_1$. Using Koszul homology, we have for all $i \geq 0$:

$$\operatorname{Tor}_i^R(k, A) = H_i(l, l_1, \ldots, l_n; A) ,$$

and

$$\operatorname{Tor}_i^S(k, A) \cong H_i(l_1, \ldots, l_n; A) .$$

Moreover, the Koszul complexes give the following exact sequences for all i:

$$\begin{aligned}0 &\to H_i(l_1, \ldots, l_n; A) \to H_i(l, l_1, \ldots, l_n; A)\\ &\to H_{i-1}(l_1, \ldots, l_n; A)(-1) \to 0\,.\end{aligned}$$

Since P is contained in the annihilator of these homology modules we see that the exact sequence splits. This shows (ii). $\square$

Proof of Theorem C. (i) Consider $(n+1)$ points of X, say $P_0, \ldots, P_n$, spanning $\mathbb{P}^n$. Let I' be the defining ideal of these points. We still have one point, say P_{n+1}, defined by the ideal J. Since the points $P_0, \ldots, P_n$ are in general position we obtain for $A' := R/I'$ (see, e.g. [23, §3])

$$\operatorname{rank}_k[H^1_P(A')]_p = \begin{cases} 0 & \text{for } p \geq 1\,,\\ n & \text{for } p = 0\,,\\ n+1 & \text{for } p \leq -1\,,\end{cases}$$

and I' is generated by forms of degree 2. We set $I := I(X)$. Since $I'/I \cong J + I'/J$ and $I' \not\subseteq J$ we have $J + I' = J + qR$ where q is a form of degree 2 of R, that is, $I'/I \cong R/J(-2)$. Hence we get the exact sequence $0 \to R/J(-2) \to A \to A' \to 0$. This gives the exact sequence

$$0 \to H^1_P(R/J)(-2) \to H^1_P(A) \to H^1_P(A') \to 0\,.$$

Since

$$\operatorname{rank}_k[H^1_P(R/J)]_p = \begin{cases} 0 & \text{for } p \geq 0\,,\\ 1 & \text{for } p \leq -1\,,\end{cases}$$

we obtain our assertion (i) of the exact sequence.

(ii)(a) First we assume that the $(n+2)$ points, say $P_0, \ldots, P_{n+1}$ are in general position in $\mathbb{P}^n$. Choose a linear form $l \in [R]_1$ such that $l(P_i) \neq 0$ for $i = 0, \ldots, n+1$, and forms $l_i \in [R]_1$ for $i = 0, \ldots, n$ such that $l_i(P_j) = 0$ for $j \in \{0, \ldots, n\} \backslash \{i\}$ and $l_i(P_i) \neq 0$. We will show that A/lA is a Gorenstein k-algebra. Hence A is Gorenstein since l is a nonzerodivisor on A.

First we have

$$\operatorname{rank}_k[A/lA]_p = \begin{cases} 0 & \text{for } p \neq 0, 1, 2\,,\\ 1 & \text{for } p = 0, 2\,,\\ n & \text{for } p = 1\,.\end{cases} \tag{1}$$

Consider a linear form $a \in [R]_1$ such that $P \cdot \bar{a} = 0$ where $\bar{a}$ is the residue class of a in A/lA.

Claim. $\bar{a} = 0$.

Proof. There are linear forms $s_0, \ldots, s_n \in [R]_1$ such that $l_i \cdot a - l \cdot s_i \in I$ for $i = 0, \ldots, n$. Take an $i \in \{0, \ldots, n\}$. Then we have for all $j = 0, \ldots, n$ with $j \neq i$: $0 = (l_i a - ls_i)(P_j) = l_i(P_j)a(P_j) - l(P_j)s_i(P_j) = -l(P_j)s_i(P_j)$. Hence $s_i(P_j) = 0$ and we get $s_i = \alpha_i l_i$ with $\alpha_i \in k$ because l_i is unique up to a nonzero constant factor. Now we have $l_i(P_{n+1}) \neq 0$ since otherwise $\{P_0, \ldots, P_{n+1}\} \setminus \{P_i\} \subseteq V(l_i)$ which contradicts the fact that $\{P_0, \ldots, P_{n+1}\}$ are in general position. Therefore, we get

$$0 = (l_i a - ls_i)(P_{n+1}) = l_i(P_{n+1})(a(P_{n+1}) - \alpha_i l(P_{n+1})) .$$

Hence $\alpha_i = a(P_{n+1})/l(P_{n+1}) =: \alpha$ which does not depend on i, that is, $s_i = \alpha l_i$. We therefore obtain $0 = (l_i a - ls_i)(P_i) = l_i(P_i)(a - \alpha l)(P_i)$, that is, $(a - \alpha l)(P_i) = 0$ for all $i = 0, \ldots, n$. Hence we have $a - \alpha l = 0$, that is, $\bar{a} = 0$. This shows our claim.

The claim now gives $[0 :_{A/lA} P]_p = 0$ for all $p \leq 1$. Since $[A/lA]_q = 0$ for all $q \geq 3$ we also have $[0 :_{A/lA} P]_p = 0$ for all $p \geq 3$. Using (1) we therefore get $[0 :_{A/lA} P]_2 = [A/lA]_2 \cong k$. Hence $0 :_{A/lA} P \cong k(-2)$, that is, A/lA is a Gorenstein k-algebra.

(b) We now assume that A is Gorenstein. We suppose $P_0, \ldots, P_{n+1}$ are not in general position in $\mathbb{P}^n$ and look for a contradiction. Let q be the smallest integer ≥ 1 such that there are $(q+2)$ points of X, spanning $\mathbb{P}^q$. Our assumption shows $q < n$. These points, spanning $\mathbb{P}^q$, are given by, say $\{P_0, \ldots, P_{q+1}\}$. Let I' again be the ideal defined by $\{P_0, \ldots, P_n\}$ and J the ideal given by $\{P_{n+1}\}$. Set $A' := A/I'$. Since $q < n$ the ideal I' contains a linear form not lying in J. Therefore we have

$$I'/I \cong J + I'/J = P/J \cong R/J(-1) ,$$

and the following exact sequence:

$$0 \to R/J(-1) \to A \to A' \to 0 . \tag{2}$$

Take a linear form $l \in [R]_1$ such that $l(P_i) \neq 0$ for all $i = 0, \ldots, n+1$. Then l is a nonzerodivisor on A, and l is also a nonzerodivisor on R/J and A'. We therefore get the following exact sequence by using $R/(J + lR) = k$:

$$0 \to k(-1) \xrightarrow{\varphi} A/lA \to A'/lA' \to 0 .$$

Hence the element $\varphi(1)$ of degree 1 in A/lA has the property that $P \cdot \varphi(1) = 0$. On the other hand, we always have $P[A/lA]_2 = 0$ but $[A/lA]_2 \neq 0$ by applying

(1). Therefore, we get $\mathrm{length}(0:_{A/lA} P) \geq 2$, that is we have no isomorphism $0:_{A/lA} P \cong k(-e)$ for all $e \in \mathbb{N}$. Hence A/lA is not a Gorenstein algebra, that is, A is not Gorenstein. This is a contradiction. This completes the proof of (ii).

(iii) We induct on $n - q \geq 0$.

Case 1: $n - q = 0$. Then it follows from (ii) that A is a Gorenstein k-algebra. Take a linear form l with $0:_A l = 0$. Then the exact sequence

$$0 \to A(-1) \xrightarrow{l} A \to A/lA \to 0$$

gives the following exact sequence:

$$0 \to A/lA \to H_P^1(A)(-1) \xrightarrow{l} H_P^1(A) \to 0 .$$

Since $((A/lA)(2))^{\vee} = (A/lA)^{\vee}(-2) \cong A/lA$, we get the exact sequence

$$0 \to (H_P^1(A)(2))^{\vee} \xrightarrow{l} (H_P^1(A)(1))^{\vee} \to A/lA \to 0 .$$

Since $0 \to A(-1) \to A \to A/lA \to 0$ is a free resolution of A/lA (as an A-module) we obtain an epimorphism $A \to (H_P^1(A)(1))^{\vee}$. Studying this epimorphism with respect to each degree we see that we indeed have an isomorphism

$$H_P^1(A) \cong A^{\vee}(-1) .$$

(We note that this is a general fact: If A is a d-dimensional graded Gorenstein k-algebra, then $H_P^d(A) \cong A^{\vee}(1 - s)$ where s is the so-called index of regularity of A, see, e.g., [20, Theorem 0.4.14]. In our situation we have $s = 2$.) Since depth $A = 1$ we consider a minimal free resolution of A as follows:

$$0 \to F_n \to \cdots \to F_1 \to R \to A \to 0 ,$$

where $F_1, \ldots, F_n$ are free R-modules. Applying the functor $\mathrm{Hom}_R(\cdot, R)$ we get the following complex:

$$0 \to R \to F^{(1)} \to \cdots \to F^{(n)} \to 0 ,$$

where $F^{(i)} := \mathrm{Hom}_R(F_i, R)$ with cohomology $\mathrm{Ext}_R^i(A, R)$. Since $\mathrm{Ext}_R^i(A, R) = 0$ for $i < n$ we obtain the following minimal free resolution of $\mathrm{Ext}_R^n(A, R)$:

$$0 \to R \to F^{(1)} \to \cdots \to F^{(n)} \to \mathrm{Ext}_R^n(A, R) \to 0 .$$

Since $\mathrm{Ext}_R^n(A, R) \cong H_P^1(A)^{\vee}(n + 1) \cong A(n + 2)$ we have therefore for $i = 1, \ldots, n - 1$:

$$F_i \cong F^{(n-i)}(-n - 2) , \qquad F_n \cong R(-n - 2) \tag{3}$$

(and, of course, $R \cong F^{(n)}(-n-2) = \mathrm{Hom}_R(F_n, R)(-n-2) = \mathrm{Hom}_R(F_n(n+2), R))$. We set for $i = 1, \ldots, n-1$, $F_i := \bigoplus_{j=1}^{\beta_i} R(-e_{ij})$ with $e_{i1}, \ldots, e_{i\beta_i} \in \mathbb{Z}$, and without loss of generality $e_{i1} \leq \cdots \leq e_{i\beta_i}$. First we have $e_{1j} \geq 2$ for all $j = 1, \ldots, \beta_1$ since $I(X)$ is generated by forms of degree ≥ 2. Moreover, we get for $1 \leq i_1 \leq i_2 \leq n-1$

$$e_{i_2 j} \geq i_2 - i_1 + e_{i_1 1} \quad \text{for all } j = 1, \ldots, \beta_{i_2}.$$

Since $F^{(n-i)}(-n-2) = \mathrm{Hom}_R(F_{n-i}, R)(-n-2) \cong \bigoplus_{j=1}^{\beta_{n-i}} R(-(n+2-e_{n-i,j}))$ we get from (3): $\beta_{n-i} = \beta_i$, and $e_{ij} = n + 2 - e_{n-i,\beta_i - j+1}$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, \beta_i$. Suppose $1 \leq i < n/2$, then we get from the above inequalities:

$$e_{i1} \leq e_{n-i,1} + 2i - n = n + 2 - e_{i,\beta_i} + 2i - n,$$

thus $e_{i1} + e_{i,\beta_i} \leq 2(i+1)$. Furthermore, we have for $j = 1, \ldots, \beta_i$, $e_{ij} \geq i - 1 + e_{11} \geq i + 1$. Hence it follows $e_{ij} = i + 1$ for all $j = 1, \ldots, \beta_i$, and $e_{n-i,j} = n - i + 1$ for all $j = 1, \ldots, \beta_i = \beta_{n-i}$. Assume $n \not\equiv 0$ (2) then we get

$$F_i = R^{\beta_i}(-i-1) \quad \text{for all } i = 1, \ldots, n-1.$$

If $n \equiv 0$ (2), then we have the same for all $i \in \{1, \ldots, n-1\} \backslash \{n/2\}$. Let $i = n/2$. Then $e_{n/2,j} = n + 2 - e_{n/2,\beta_{n/2}-j+1}$, thus $e_{n/2,1} + e_{n/2,\beta_{n/2}} = n + 2$. Hence we also get

$$F_{n/2} \cong R^{\beta_{n/2}}(-n/2 - 1).$$

Studying the resolution of A with respect to the degrees $2, 3, \ldots, n$ we obtain for $i = 1, \ldots, n-1$

$$\beta_i = \sum_{j=1}^{i-1} (-1)^{j-1} \binom{n+j}{j} \beta_{i-j} + (-1)^{i-1} \binom{n+i+1}{i+1} + (-1)^i (n+2),$$

since $\mathrm{rank}_k [A]_{i+1} = n + 2$ for $i \geq 1$. Using induction on i we therefore obtain

$$\beta_i = (n+2)\binom{n}{i} - \binom{n+2}{i+1}$$

by applying the relationship

$$\sum_{j=0}^{i} (-1)^j \binom{x+j}{j} \binom{y}{i-j} = (-1)^i \binom{x-y+i}{i}.$$

This shows Case 1, that is, $n - q = 0$.

Case 2: $n - q > 0$. Using our notations of the proof of (ii)(b) we consider again the exact sequence (2). Let l' be the linear form of I' not lying in J. We set $S := R/l' \cdot R$ and we get from (2) the long exact sequence:

$$\cdots \to \operatorname{Tor}_{i+1}^R(k, A') \xrightarrow{\partial_{i+1}} \operatorname{Tor}_i^R(k, R/J)(-1) \to \operatorname{Tor}_i^R(k, A)$$
$$\to \operatorname{Tor}_i^R(k, A') \xrightarrow{\partial_i} \operatorname{Tor}_{i-1}^R(k, R/J)(-1) \to \cdots .$$

Induction and Lemma 2 give for $i \geq 2$

$$[\operatorname{Tor}_i^R(k, A')]_p \cong [\operatorname{Tor}_i^S(k, A')]_p \oplus [\operatorname{Tor}_{i-1}^S(k, A')]_{p-1} = 0$$

for all $p \neq i+1, i+2$. Since

$$[\operatorname{Tor}_{i-1}^R(k, R/J)(-1)]_p = [\operatorname{Tor}_{i-1}^R(k, R/J)]_{p-1} = 0$$

for all $p \neq i$, we get from the above long exact sequence the following exact sequences:

$$0 \to \operatorname{Tor}_i^R(k, R/J)(-1) \to \operatorname{Tor}_i^R(k, A) \to \operatorname{Tor}_i^R(k, A') \to 0$$

for all $i \geq 2$, and

$$0 \to \operatorname{Tor}_1^R(k, R/J)(-1) \to \operatorname{Tor}_1^R(k, A)$$
$$\to \operatorname{Tor}_1^R(k, A') \to k(-1) \to k \to k \to 0 .$$

It follows from Lemma 2 that $\operatorname{Tor}_1^R(k, A') \cong \operatorname{Tor}_1^S(k, A') \oplus k(-1)$, and by induction we have for all $p \neq 2, 3$: $[\operatorname{Tor}_1^S(k, A')]_p = 0$. Hence we obtain the exact sequence

$$0 \to \operatorname{Tor}_1^R(k, R/J)(-1) \to \operatorname{Tor}_1^R(k, A) \to \operatorname{Tor}_i^S(k, A') \to 0 .$$

Applying Lemma 2 we therefore get for $i \geq 2$

$$\operatorname{Tor}_i^R(k, A) \cong k^{\binom{n}{i}}(-i-1) \oplus \operatorname{Tor}_i^S(k, A') \oplus \operatorname{Tor}_{i-1}^S(k, A')(-1) ,$$

and

$$\operatorname{Tor}_1^R(k, A) \cong k^n(-2) \oplus \operatorname{Tor}_1^S(k, A') .$$

(Note that $\operatorname{Tor}_i^R(k, R/J)(-1) \cong k^{\binom{n}{i}}(-i-1)$.) By induction we therefore obtain for $i = 1, \ldots, n$ the desired Betti numbers. This completes the proof of Theorem C. □

4. Castelnuovo's regularity and the Hilbert polynomial

Before embarking on the proof of Theorems A and B, we need a careful study of Castelnuovo's regularity and of the Hilbert polynomial of subschemes with degree = codimension + 2. The aim of this section is therefore to prove the following theorem.

Let k be an algebraically closed field of arbitrary characteristic.

Theorem 3. *Let X be a pure-dimensional, reduced and nondegenerate subscheme of $\mathbb{P}_k^n$ which is connected in codimension* 1 *if* $\dim X \geq 1$. *Let $I(X)$ be the defining ideal of X in $R := k[x_0, \ldots, x_n]$. Let A be the graded k-algebra $R/I(X)$ of* $\dim A =: d$ *and* $\operatorname{depth} A =: r \geq 1$. *Assume that* $\operatorname{degree}(X) = \operatorname{codim}(X) + 2$. *Then we have*:

(i) $\operatorname{reg} A = 2$, *where* $\operatorname{reg} A$ *denotes Castelnuovo's regularity of* A.

(ii) $$h(t, A) = (n + 3 - d)\binom{t + d - 2}{d - 1} + \binom{t + d - 2}{d - 2} - \binom{t + r - 3}{r - 2},$$

where $h(t, A)$ is the Hilbert polynomial of A and we set $\binom{x}{-1} := 0$.

Before proving this theorem we have to prove several preliminary results.

Lemma 4. *Let A be a pure-dimensional and reduced graded k-algebra with* $\dim A \geq 2$ *such that* $\operatorname{Proj} A$ *is connected in codimension* 1. *Let l be a generic linear form of A. We set $B := A/l \cdot A : \langle P \rangle$. Then B is again a pure-dimensional and reduced graded k-algebra with the following properties*: $\dim B = \dim A - 1$, $\operatorname{codim} B = \operatorname{codim} A$, $h_0(B) = h_0(A)$, $\operatorname{depth} B = \max\{1, \operatorname{depth} A - 1\}$ *and* $\operatorname{rank}_k[B]_1 = \operatorname{rank}_k[A]_1 - 1$. *Moreover, if* $\dim B \geq 2$, *then* $\operatorname{Proj} B$ *is also connected in codimension* 1.

Proof. It follows from Satz 5.2 of [4] that B is reduced (and pure-dimensional). The assertions $\dim B = \dim A - 1$ and $h_0(B) = h_0(A)$ are trivial. If $\dim B \geq 2$, then $\operatorname{Proj} B$ is connected in codimension 1 by using Grothendieck's connectedness theorem [10, Exp. 13]. Of course, $\operatorname{depth} B \geq 1$. Assume first $\operatorname{depth} A = 1$. We suppose $\operatorname{depth} B \geq 2$ and look for a contradiction. Considering the exact sequence $0 \to A(-1) \to A \to A/lA \to 0$ and using $H_P^1(A/lA) \cong H_P^1(B) = 0$ we get the following exact sequence:

$$H_P^0(A/lA) \to H_P^1(A)(-1) \xrightarrow{l} H_P^1(A) \to 0\,.$$

It follows from Lemma 1 that $H_P^1(A)$ is a noetherian A-module. This implies by Nakayama's Lemma that $H_P^1(A) = 0$. Hence we have a contradiction to $\operatorname{depth} A = 1$.

We now assume $r := \operatorname{depth} A \geq 2$. Then the exact sequence $0 \to A(-1) \xrightarrow{l} A \to A/lA \to 0$ gives the exact sequence

$$0 \to H_P^{r-1}(B) \to H_P^r(A)(-1) \xrightarrow{l} H_P^r(A)$$

since $H_P^i(A/lA) \cong H_P^i(B)$ for $i \geq 1$.

Assume $H_P^{r-1}(B)=0$. Then the map

$$[H_P^r(A)]_{p-1} \xrightarrow{l} [H_P^r(A)]_p$$

is injective for all $p \in \mathbb{Z}$. Since $[H_P^r(A)]_p = 0$ for p large enough we get $H_P^r(A) = 0$. This gives immediately a contradiction to $r = \text{depth } A$. Since $H_P^i(B) = 0$ for $i \leq r-2$ we get depth $B = r - 1 = \max\{1, \text{depth } A - 1\}$. Finally, the theorem of Bertini of [5, Theorem 4.11] provides $\text{rank}_k[B]_1 = \text{rank}_k[A]_1 - 1$. This proves Lemma 4. □

Remark. Let the situation be as described in Lemma 4. Let $l_1, \ldots, l_i$ be generic linear forms of A with $0 \leq i < \dim A$. We set $B := A/(l_1, \ldots, l_i)A : \langle P \rangle$. Then it follows from Lemma 4 that B is reduced and pure-dimensional. Moreover, we have $\text{rank}_k[B]_1 = \text{rank}_k[A]_1 - i$. If $\dim B \geq 2$, then Proj B is connected in codimension 1.

Lemma 4 and Theorem C yield us the possibility to give a straightforward proof of a well-known characterization of varieties of minimal degree (see also [2]):

Theorem 5. *Let X be a pure-dimensional, reduced and nondegenerate subscheme of $\mathbb{P}_k^n$ being connected in codimension* 1 *if* $\dim X \geq 1$. *Let $I(X)$ be the defining ideal of X in $R = k[x_0, \ldots, x_n]$. Then the following conditions are equivalent*:

(i) degree(X) = codim$(X) + 1$.

(ii) *X is arithmetically Cohen–Macaulay and either Castelnuovo's regularity* reg $R/I(X) = 1$ *or* $R/I(X) \cong k[x_1, \ldots, x_d]$ *with indeterminates* $x_1, \ldots, x_d$, $d := \dim R/I(X)$.

Proof. First we show (i)$\Rightarrow$(ii). We induct on d. Let $d = 1$. Hence $I(X)$ is the ideal of $(n+1)$ points in $\mathbb{P}^n$ being in general position (see Section 2). If $n = 0$, then $I(X) = 0$, that is $R/I(X) \cong k[x_0]$. Let $n \geq 1$ and $A = R/I(X)$. Then it follows that $[H_P^1(A)]_p = 0$ for all $p \geq 1$ and $[H_P^1(A)]_0 \neq 0$ (see [23, Theorem 1(i) and Corollary 1]), and of course, $H_P^0(A) = 0$. Hence we get reg $A = 1 + e(H_P^1(A)) = 1 + 0 = 1$. Since X is pure-dimensional we also have the Cohen–Macaulay property.

Let $d \geq 2$. Let B be as in Lemma 4. This lemma shows that we can apply induction. Hence B is Cohen–Macaulay, that is, $H_P^i(B) = 0$ for all $i \neq d-1$. Since $H_P^i(B) \cong H_P^i(A/lA)$ for $i \geq 1$ we therefore obtain that $H_P^{i+1}(A) = 0$ for all $i \neq 0, d-1$, and that the sequence $H_P^1(A)(-1) \xrightarrow{l} H_P^1(A) \to 0$ is exact. Since $H_P^1(A)$ is noetherian by Lemma 1 it follows $H_P^1(A) = 0$ by Nakayama's Lemma. Therefore, A is Cohen–Macaulay. Since now $B = A/lA$, we have reg A = reg A/lA = reg B. If reg $B = 1$, then we are done. If $B \cong k[y_1, \ldots, y_{d-1}]$, then $A \cong k[x_1, \ldots, x_d]$ since $B = A/lA$.

(i)$\Leftarrow$(ii). If $A \cong k[x_1, \ldots, x_d]$, then A is of minimal degree. This shows (i).

Assume that A is Cohen–Macaulay and $\operatorname{reg} A = 1$. We then induct on d. Let $d = 1$. We suppose that $h_0(A) \geq n+2$ and look for a contradiction. Let I' be the ideal of $(n+2)$ points, spanning $\mathbb{P}^n_k$ such that $I(X) \subseteq I'$. We thus have an epimorphism $\pi : R/I \to R/I'$. Since $\dim \operatorname{Ker} \pi \leq 1$, we get an epimorphism

$$H^1_P(R/I) \to H^1_P(R/I') \to 0 .$$

Theorem C(i) shows that $[H^1_P(R/I')]_1 \cong k$, hence $[H^1_P(A)]_1 \neq 0$. This immediately gives a contradiction to $\operatorname{reg} A = 1$. This shows the case $d = 1$. Let $d \geq 2$. Take B as in Lemma 4. It follows by induction that B is of minimal degree. Lemma 4 provides this also for A. This completes the proof of Theorem 5. □

Lemma 6. *Let A be a pure-dimensional and reduced graded k-algebra of* $\dim A =: d \geq 1$. *If $d \geq 2$, then we assume that* Proj A *is connected in codimension* 1. *We set* $r := \operatorname{depth} A$. *Suppose that* $h_0(A) = \operatorname{codim} A + 2$ (*that is,* $h_0(A) = \operatorname{rank}_k[A]_1 + 2 - d$), *then we have*:

(i) $H^i_P(A) = 0$ *for all* $i \neq r, d$.

(ii) *If* $r < d$, *then we get*

$$[H^r_P(A)]_p = 0 \quad \textit{for all } p \geq 3 - r , \qquad [H^r_P(A)]_{2-r} \cong k ,$$

$$[H^d_P(A)]_p = 0 \quad \textit{for all } p \geq 2 - d ,$$

and

$$[H^d_P(A)]_{1-d} \cong k^{n+2-d} \quad \textit{in assuming } r \leq d-2 .$$

(*Note that by our general assumption* $n = \operatorname{rank}_k[A]_1 - 1$.)

(iii) *If* $r = d$, *then we obtain*

$$[H^d_P(A)]_p = 0 \quad \textit{for all } p \geq 3 - d , \qquad [H^d_P(A)]_{2-d} \cong k .$$

Proof. We induct on d. Let $d = 1$. In this case Lemma 6, that is, the assertion (iii) follows from our Theorem C(i). Let $d \geq 2$. Consider a generic linear form l of $R = k[x_0, \ldots, x_n]$, and we set $B := A/lA : \langle P \rangle$. Our Lemma 4 shows that we can apply induction to B. Moroever, we have $H^i_P(B) \cong H^i_P(A/lA)$ for $i \geq 1$. Consider the exact sequence $0 \to A(-1) \xrightarrow{l} A \to A/lA \to 0$. First we assume $r \geq 2$. Then $B = A/lA$ and we have exact sequences

$$H^{i-1}_P(A/lA) \to H^i_P(A)(-1) \xrightarrow{l} H^i_P(A) .$$

Hence we get for all $i \neq r, d$ monomorphisms

$$H^i_P(A)(-1) \xrightarrow{l} H^i_P(A)$$

(since depth $A/lA = r-1$, dim $A/lA = d-1$ and, by induction, $H_P^{i-1}(A/lA) = 0$ for all $i \neq r, d$). This gives monomorphisms $[H_P^i(A)]_{p-1} \to [H_P^i(A)]_p$ and therefore $H_P^i(A) = 0$ since $[H_P^i(A)]_p = 0$ for $p \gg 0$. This shows (i) in assuming $r \geq 2$.

Let $r = 1$. Then the same argument, however, provides $H_P^i(A) = 0$ for all $i \neq 1, 2, d$. Moroever, we have the exact sequence

$$H_P^1(A)(-1) \xrightarrow{l} H_P^1(A) \xrightarrow{\varphi} H_P^1(B) \xrightarrow{\psi} H_P^2(A)(-1) \xrightarrow{l} H_P^2(A) . \tag{4}$$

If $d \geq 3$, then dim $B \geq 2$ and depth $B = 1$ by Lemma 4. Consider the exact sequence

$$0 \to B \to H^0(B) \to H_P^1(B) \to 0$$

(see Section 2). It follows from Lemma 4 and Lemma 1 that $[H_P^1(B)]_p = 0$ for all $p \leq 0$. By induction we have $[H_P^1(B)]_q = 0$ for all $q \geq 2$, and $[H_P^1(B)]_1 \cong k$. Hence $H_P^1(B) \cong k(-1)$. Therefore we see that either $\varphi = 0$ or $\psi = 0$. Assume that $\varphi = 0$. Using (the graded version of) Nakayama's Lemma we then get $H_P^1(A) = 0$ since $H_P^1(A)$ is noetherian. This is a contradiction. hence we have $\varphi \neq 0$ and $\psi = 0$. This gives an exact sequence

$$H_P^1(A)(-1) \xrightarrow{l} H_P^1(A) \to k(-1) \to 0 \tag{5}$$

and a monomorphism $H_P^2(A)(-1) \xrightarrow{l} H_P^2(A)$ and we again obtain as above that $H_P^2(A) = 0$. This shows (i) in case $d \geq 3$ and $r = 1$. If $d = 2$ and $r = 1$, then (i) is trivial. This completes the proof of (i).

We show (ii). First we assume $d = 2$, $r = 1$. As above we have $[H_P^1(A)]_p = 0$ for all $p \leq 0$. Theorem C provides $[H_P^1(B)]_q = 0$ for all $q \geq 2$, and $[H_P^1(B)]_1 \cong k$. Hence in (4) we have $[\varphi]_i = 0$ for all $i \neq 1$. Since $H_P^1(A) \neq 0$ is again noetherian we get $[\varphi]_1 \neq 0$. Hence $[\psi]_i = 0$ for all $i \geq 1$. This gives monomorphisms

$$[H_P^2(A)]_j \xrightarrow{l} [H_P^2(A)]_{j+1} \quad \text{for } j \geq 0 ,$$

that is, $[H_P^2(A)]_p = 0$ for all $p \geq 0$.

Since $[\varphi]_0 = 0$ we have that $[\psi]_0$ is an isomorphism, that is, $[H_P^2(A)]_{-1} \cong k^n$ by Theorem C(i).

On the other hand, we have in this case the exact sequence

$$H_P^1(A)(-1) \xrightarrow{l} H_P^1(A) \to k(-1) \to 0$$

which is the same as the above sequence (5) given in case $d \geq 3$.

If $d \geq 3$, then we also have the exact sequence

$$0 \to H_P^{d-1}(B) \to H_P^d(A)(-1) \xrightarrow{l} H_P^d(A) \to 0 .$$

Using induction we therefore obtain

$$[H_P^d(A)]_p = 0 \quad \text{for all } p \geq 2 - d ,$$

$$[H_P^d(A)]_{1-d} \cong [H_P^{d-1}(B)]_{2-d} \cong k^{n-1+2-(d-1)} = k^{n+2-d} .$$

Hence we still have to show in case $d \geq 2$, $r = 1$: $[H_P^1(A)]_p = 0$ for all $p \geq 2$, and $[H_P^1(A)]_1 \cong k$. Therefore we will prove the following claim which also extends the key result (see [21, Lemma 3(ii)]) in proving Theorem 2 of [21].

Claim. *Assume that $d \geq 2$ and $r = 1$. Then we have $H_P^1(A) \cong k(-1)$.*

Proof. First we obtain $[H_P^1(A)]_q = 0$ for all $q \leq 0$ by Lemma 1. Hence we get from the sequence (5) that $[H_P^1(A)]_1 \cong k$. We study now the exact sequence (see Section 2).

$$0 \to A \to H^0(A) \to H_P^1(A) \to 0 . \tag{6}$$

Since

$$H_P^i(H^0(A)) \cong \begin{Bmatrix} 0 & \text{for } i = 0, 1 , \\ H_P^i(A) & \text{for } i \geq 2 , \end{Bmatrix}$$

and $d \geq 2$ we obtain $H_P^i(H^0(A)) = 0$ for all $i \neq d$ and $H_P^d(H^0(A)) \cong H_P^d(A)$. Therefore, we get for Castelnuovo's regularity of $H^0(A)$:

$$\operatorname{reg} H^0(A) = d + e(H_P^d(H^0(A))) = d + e(H_P^d(A)) \leq 1 ,$$

since $[H_P^d(A)]_p = 0$ for all $p \geq 2 - d$ (see above).

Furthermore, we obtain from (5) an isomorphism $H_P^1(A) \otimes k \to k(-1) \otimes k \cong k(-1)$, that is, we have $H_P^1(A) \cong R/\mathfrak{q}(-1)$ (as R-modules) where $\mathfrak{q}$ is a primary ideal belonging to P (note that $\operatorname{length}(H_P^1(A)) < \infty$). Consider the embedding $A \hookrightarrow H^0(A)$ of (6). Since the identity element 1 of A belongs to the set of generators of $H^0(A)$ we have a monomorphism $A \otimes k \to H^0(A) \otimes k$. Therefore, we get the following epimorphism

$$\operatorname{Tor}_1^R(k, H^0(A)) \to \operatorname{Tor}_1^R(k, H_P^1(A)) .$$

We proved above that $\operatorname{reg} H^0(A) \leq 1$. Hence $\operatorname{Tor}_1^R(k, H^0(A))$ is generated by elements of degree ≤ 2, and therefore also $\operatorname{Tor}_1^R(k, H_P^1(A))$. Since $H_P^1(A) \cong R/\mathfrak{q}(-1)$ we now have:

$$\operatorname{Tor}_1^R(k, H_P^1(A)) \cong \operatorname{Tor}_1^R(k, R/\mathfrak{q})(-1) \cong (k \otimes \mathfrak{q})(-1) \cong \mathfrak{q}/P \cdot \mathfrak{q}(-1),$$

that is, $\mathfrak{q}$ is generated by elements of degree ≤ 1. Hence we get $\mathfrak{q} = P$, that is, $H_P^1(A) \cong k(-1)$. This shows our claim.

We now assume $2 \leq r < d$. Consider the exact sequences

$$0 \to H_P^{r-1}(B) \to H_P^r(A)(-1) \xrightarrow{l} H_P^r(A),$$

and

$$H_P^{d-1}(B) \to H_P^d(A)(-1) \xrightarrow{l} H_P^d(A) \to 0.$$

These sequences provide (ii) by induction.

Let $r = d \geq 2$. Then consider

$$0 \to H_P^{d-1}(B) \to H_P^d(A)(-1) \xrightarrow{l} H_P^d(A) \to 0$$

using induction we also get (iii). This proves our Lemma 6. $\square$

Remark. Lemma 6 shows that $[H_P^d(A)]_{1-d} \cong k^{n+2-d}$ for $r \leq d-2$. Moreover, this result is also true in case $r = d-1$. This follows from the proof of Lemma 6 by applying Theorem B(i).

Proof of Theorem 1. (i) The Castelnuovo regularity results immediately from Lemma 6.

(ii) We induct on d. The case $d = 1$ is trivial. Let $d \geq 2$. Take a generic linear form $l \in [A]_1$ and set $B := A/lA : \langle P \rangle$. We have the well-known property for the Hilbert polynomial

$$h(t, B) = h(t, A/lA) = h(t, A) - h(t-1, A).$$

Case 1: $r \geq 2$. Then we obtain for $t > 0$:

$$\begin{aligned}
h(t, A) &= h(0, A) + \sum_{i=1}^{t} h(i, B) \quad \text{(by induction and Lemma 4)} \\
&= h(0, A) + \sum_{i=1}^{t} \left((n+3-d)\binom{i+d-3}{d-2} \right. \\
&\qquad \left. + \binom{i+d-3}{d-3} - \binom{i+r-4}{r-3} \right) \\
&= h(0, A) + (n+3-d)\binom{t+d-2}{d-1} + \binom{t+d-2}{d-2} \\
&\quad - 1 - \begin{cases} \binom{t+r-3}{r-2} & \text{for } r \geq 3, \\ 0 & \text{for } r = 2, \end{cases} \\
&= (n+3-d)\binom{t+d-2}{d-1} + \binom{t+d-2}{d-2} - \binom{t+r-3}{r-2} \\
&\quad + h(0, A) + \begin{cases} -1 & \text{for } r \geq 3, \\ 0 & \text{for } r = 2. \end{cases}
\end{aligned}$$

Using [18, Chapter 3, §6] (see also [20, p. 100]) we get

$$\begin{aligned}
h(0, A) &= \sum_{i\geq 0} (-1)^i \operatorname{rank}_k[H^i(A)]_0 \\
&= 1 + \sum_{i\geq 2} (-1)^{i-1} \operatorname{rank}_k[H^i(A)]_0 \\
&= \begin{cases} 1 + (-1)^{r-1} \operatorname{rank}_k[H^r_P(A)]_0 + (-1)^d \operatorname{rank}_k[H^d_P(A)]_0 \\ \quad \text{for } r < d\,, \\ 1 + (-1)^{d-1} \operatorname{rank}_k[H^d_P(A)]_0 \\ \quad \text{for } r = d \end{cases} \\
&= \begin{cases} 1 & \text{for } r \geq 3 \\ 0 & \text{for } r = 2 \end{cases}
\end{aligned}$$

by applying Lemma 6(ii) and (iii), respectively. This shows Case 1.

Case 2: $r = 1$. Then depth $B = 1$, and we have again

$$\begin{aligned}
h(t, A) &= h(0, A) + \sum_{i=1}^{t} h(i, B) \quad \text{(by induction)} \\
&= h(0, A) + \sum_{i=1}^{t} \left((n + 3 - d)\binom{i + d - 3}{d - 2} + \binom{i + d - 3}{d - 3}\right) \\
&= h(0, A) + (n + 3 - d)\binom{t + d - 2}{d - 1} + \binom{t + d - 2}{d - 2} - 1 \\
&= (n + 3 - d)\binom{t + d - 2}{d - 1} + \binom{t + d - 2}{d - 2}
\end{aligned}$$

since by Lemma 6(ii) we have again $h(0, A) = \operatorname{rank}_k[H^0(A)]_0 = 1$. This gives Case 2 and completes the proof of Theorem 3. □

5. Proof of Theorem B

Before proving Theorem B we state a seemingly technical yet in the sequel very useful result.

Lemma 7. *Let A be a graded k-algebra. Let N be a finitely generated graded A-module with* Krull-dim $N \geq 1$. *Let l be a generic form of $[A]_1$. Assume that there is a prime ideal $\mathfrak{p}$ of A such that $N/l \cdot N \cong A/\mathfrak{p}$. Then there is a prime ideal $\mathfrak{q}$ of A with $N \cong A/\mathfrak{q}$.*

Proof. Since $N \otimes k \cong (N/l \cdot N) \otimes k \cong A/\mathfrak{p} \otimes k$ we have $N \cong A/\mathfrak{q}$ where $\mathfrak{q}$ is an ideal of A such that $\mathfrak{p} = \mathfrak{q} + l \cdot A$. If $\mathfrak{p} = P$, then $\mathfrak{q}$ is a prime ideal since $\dim A/\mathfrak{q} = \dim N \geq 1$. Suppose that $\mathfrak{p} \neq P$ for generic $l \in [A]_1$. Choose generic elements $l_1, l_2, \ldots$ of $[A]_1$. Then $l_{i+1} \not\subseteq (\mathfrak{q} + l_1 A) \cup \cdots \cup (\mathfrak{q} + l_i A)$ and $\mathfrak{q} + l_i A$ is a prime ideal for all $i \geq 1$. Let M be a finite subset of $\mathbb{N}^+$ (the integers ≥ 1). Then it follows

$$\bigcap_{i \in M} (\mathfrak{q} + l_i \cdot A) = \mathfrak{q} + \Big(\prod_{i \in M} l_i\Big) \cdot A .$$

For an element $x \in A$ we set $M_x := \{i \in \mathbb{N}^+ : x \in \mathfrak{q} + l_i A\}$.

Claim. $x \not\in \mathfrak{q}$ *if and only if* card $M_x < \infty$, *and in this case* card $M_x \leq$ *degree of* x.

Proof. If $x \in \mathfrak{q}$, then $M_x = \mathbb{N}^+$. Let $x \not\in \mathfrak{q}$. We set $p :=$ degree of x. Suppose card M_x is not finite, then we choose a subset M of M_x consisting of $(p+1)$ elements. Then we have $x \in \bigcap_{i \in M} (\mathfrak{q} + l_i A) = \mathfrak{q} + f \cdot A$ where $f = \prod_{i \in M} l_i$. Since degree$(f) = p+1$ and degree$(x) = p$ we get $x \in \mathfrak{q}$. This is a contradiction. This shows our claim.

Take elements $a, b \in A$ such that $a \cdot b \in \mathfrak{q}$ and $a \not\in \mathfrak{q}$. Then we get $b \in \mathfrak{q} + l_i A$ for all $i \in \mathbb{N}^+ \backslash M_a$. This claim gives $b \in \mathfrak{q}$, that is, $\mathfrak{q}$ is a prime ideal. $\square$

Remark. Let us describe more explicitly the prime ideal $\mathfrak{q}$. Suppose that $\mathfrak{p}$ is generated by l and elements, say $f_1, \ldots, f_r$. Then $\mathfrak{q}$ is generated by elements, say $g_1, \ldots, g_r$ such that degree$(f_i) =$ degree(g_i) for $i = 1, \ldots, r$. This follows since the generic linear form $l \not\in \mathfrak{q}$ because $\mathfrak{q} \neq P$. Hence $\mathfrak{q} \otimes k \cong (\mathfrak{q} + lA/lA) \otimes k = (\mathfrak{p}/lA) \otimes k$.

Proof of Theorem B. (i) The assertion $H_P^i(A) = 0$ for all $i \neq r, d$ follows from Lemma 6(i). Proving the structure theorem of $H_P^r(A)$ we induct on r. The case $r = 1$ follows from the claim of the proof of Lemma 6. Let $r > 1$. Take a generic element $l \in [A]_1$. Lemma 4 shows that we can apply the induction hypothesis to $B := A/lA$. Hence we consider the following exact sequence $0 \to A(-1) \xrightarrow{l} A \to B \to 0$ which gives an exact sequence

$$0 \to H_P^{r-1}(B) \to H_P^r(A)(-1) \xrightarrow{l} H_P^r(A) \to H_P^r(B) . \tag{7}$$

Dualizing and shifting degrees yield with $N := H_P^r(A)\check{}\,(r-2)$ an exact sequence

$$H_P^r(B)\check{}\,(r-3) \to N(-1) \xrightarrow{l} N \to H_P^{r-1}(B)\check{}\,(r-3) \to 0 . \tag{8}$$

By induction $H_P^{r-1}(B)\check{}\,(r-3) \cong k[z_0, \ldots, z_{r-3}]$ with algebraically independent elements $z_0, \ldots, z_{r-3}$ of $[B]_1$. If dim $N \geq 1$, Lemma 7 (together with the remark made after the proof of this lemma) and our induction hypothesis give the proof of Theorem B(i). Therefore, it is sufficient to show that dim $N \geq 1$. But this is clear if either $r \geq 3$ or $r \leq d-2$. (In the first case we have with (8)

$$\dim N \geq \dim N/lN = \dim H_P^{r-1}(B)\check{\ }(r-3) = \dim k[z_0, \ldots, z_{r-3}] \geq 1$$

and in the second case $H_P^r(B) = 0$ and therefore by (8) l is not a zerodivisor on N, that is, $\dim N \geq 1$.)

Therefore, it remains to consider the case $r = 2$, $d = 3$.

In this situation (8) gives the exact sequence

$$N(-1) \xrightarrow{l} N \to k \to 0 .$$

Therefore, $N \otimes k \cong k \otimes k = k$ and this shows that $N \cong A/\mathfrak{a}$ with an (homogeneous) ideal $\mathfrak{a} \subset A$ $(\mathfrak{a} = \operatorname{Ann} N)$. Since $A/P = k \cong N/lN \cong A/\mathfrak{a} + lA$ we have $\mathfrak{a} + lA = P$ and therefore $\operatorname{rank}_k[\mathfrak{a}]_1 \geq n$. Let $l_1, \ldots, l_n$ denote n linear independent elements of $[\mathfrak{a}]_1$. Then $(l_1, \ldots, l_n)N = 0$ and hence

$$(l_1, \ldots, l_n)H_P^2(A) = 0 .$$

Choose an element, say $l_0 \in [A]_1$, such that $l_0, l_1, \ldots, l_n$ are linearly independent, that is, $(l_0, l_1, \ldots, l_n) \cdot A = P$.

The following consideration is central to the proof in case $r = 2$, $d = 3$: Let $D_k(A)$ denote the A-module of Kähler differentials of A over k (see, e.g., [14]). We note that in our situation (A is a factor algebra of a polynomial k-algebra) $D_k(A)$ coincides with the module of universal finite k-differentials (see, e.g., [17]). Therefore we have for an arbitrary prime ideal $\mathfrak{p}$ of A by [17, Satz (5.1)]

$$\mu_\mathfrak{p}(D_k(A)) \geq \dim A_\mathfrak{p} + \dim A/\mathfrak{p} \geq \dim A/\mathfrak{p} ,$$

where $\mu_\mathfrak{p}(D_k(A))$ denotes the minimal number of generators of $D_k(A) \otimes_A A_\mathfrak{p}$ (considered as an $A_\mathfrak{p}$-module).

Moreover, we note that $D_k(A)$ is generated (as an A-module) by $d(l_0), \ldots, d(l_n)$, where $d : A \to D_k(A)$ denotes the corresponding k-derivation. Let M denote the submodule of $D_k(A)$ generated by $d(l_1), \ldots, d(l_n)$. Then we have by the above inequality for all prime ideals $\mathfrak{p}$ of A,

$$\mu_\mathfrak{p}(D_k(A)) - \mu_\mathfrak{p}(D_k(A)/M) \geq \dim A/\mathfrak{p} - 1 .$$

Therefore, we can apply Satz (1.5) of [4] (take $U := \operatorname{Spec} A$, $S := k$ and $t = 1$), and we get elements $\alpha_2, \ldots, \alpha_n \in k$ such that for $\lambda := l_1 + \alpha_2 l_2 + \cdots + \alpha_n l_n$ the element $d(\lambda) = d(l_1) + \alpha_2 d(l_2) + \cdots + \alpha_n d(l_n)$ is basic in $D_k(A)$ at every prime ideal $\mathfrak{p}$ of A with $\dim A/\mathfrak{p} \geq 2$. Moreover, λ is not contained in any minimal prime ideal of A (i.e., λ is not a zerodivisor on A). Lemma 2.2 of [4] provides that $\lambda \not\in \mathfrak{p}^{(2)}$ for all prime ideals $\mathfrak{p}$ of A with $A/\mathfrak{p} \geq 2$.

After constructing the element λ we consider the k-algebra $C := A/\lambda A$. Then $\dim C = 2$ and $\operatorname{depth} C = 1$. Let $\mathfrak{q}$ be a minimal prime ideal of λA. Since

$\dim A/\mathfrak{q} = 2$ we have $\lambda \not\in \mathfrak{q}^{(2)}$. Therefore, C is reduced at every prime ideal of dimension $=2$ (see [4, Satz (2.1)]).

Consider the exact sequence

$$0 \to A(-1) \xrightarrow{\lambda} A \to C \to 0 .$$

Since $\lambda \cdot H_P^2(A) = 0$, we get an isomorphism $H_P^1(C) \cong H_P^2(A)(-1)$.

We now assume that $\dim N = 0$, that is, $H_P^2(A)$ is a noetherian A-module and look for a contradiction. Then $H_P^1(C)$ is a noetherian C-module. Lemma 1(i) shows that C is pure-dimensional (note that depth $C = 1$). The above consideration therefore provides that C is also reduced. The connectedness theorem of Grothendieck (see [10]) shows that Proj C is connected in codimension 1. Hence Lemma 1(ii) yields $[H_P^1(C)]_p = 0$ for all $p \leq 0$. The above isomorphism gives $[H_P^2(A)]_q = 0$ for all $q \leq -1$. Now consider the exact sequence (7), i.e. in our situation the exact sequence

$$0 \to k(-1) \to H_P^2(A)(-1) \xrightarrow{l} H_P^2(A) \to H_P^2(B) ,$$

in degree zero. Then we obtain $[H_P^2(A)]_0 = 0$ since also $[H_P^2(B)]_0 = 0$ by Lemma 6(ii). Then we study this exact sequence in degree 1. It provides the exact sequence $0 \to k \to 0$ which is a contradiction. Hence Lemma 7 proves also the case $r = 2$, $d = 3$. This completes the proof of Theorem B(i).

(ii) Assume depth $A = 1$. Then Theorem B(i) shows that $H_P^1(A) \cong k(-1)$ and $H_P^i(A) = 0$ for all $i \neq 1, d$. Hence A is a Buchsbaum non-Cohen–Macaulay k-algebra (see, e.g., [20, Corollary I.3.6]).

Suppose that A is Buchsbaum and not Cohen–Macaulay. Then $H_P^r(A)$ has finite length. Therefore Theorem B(i) shows that depth $A = 1$. This provides the assertion (ii) of Theorem B.

(iii) Let $\mathfrak{p}$ be a prime ideal of A. We also consider $\mathfrak{p}$ as a prime ideal of R. Let $E(\mathfrak{p})$ be the injective hull of $R_\mathfrak{p}/\mathfrak{p} \cdot R_\mathfrak{p}$. Using local duality we get

$$\begin{aligned} \operatorname{Hom}_{R_\mathfrak{p}}(H^i_{\mathfrak{p}R_\mathfrak{p}}(A_\mathfrak{p}), E(\mathfrak{p})) &\cong \operatorname{Ext}_{R_\mathfrak{p}}^{\dim R_\mathfrak{p} - i}(A_\mathfrak{p}, R_\mathfrak{p}) \\ &\cong (\operatorname{Ext}_R^{n+1-\dim A/\mathfrak{p} - i}(A, R))_\mathfrak{p} \\ &\cong (H_P^{i+\dim A/\mathfrak{p}}(A)^\vee(n+1))_\mathfrak{p} . \end{aligned} \tag{9}$$

From this it follows that $H^i_{\mathfrak{p}R_\mathfrak{p}}(A_\mathfrak{p}) = 0$ for all $i \neq r - \dim A/\mathfrak{p}$, $d - \dim A/\mathfrak{p}$.

Our Theorem B(i) provides that $H_P^r(A) \cong (A/\mathfrak{q})^\vee(r-2)$ where $\mathfrak{q}$ is a prime ideal of A generated by $n + 2 - r$ linearly independent elements of $[A]_1$.

Suppose that $r = 2$, that is, $\dim A/\mathfrak{q} = 1$. It follows from (9) that

$$\operatorname{Hom}(H^{r-1}_{\mathfrak{q}R_\mathfrak{q}}(A_\mathfrak{q}), E(\mathfrak{q})) \cong (A/\mathfrak{q}(n+3-r))_\mathfrak{q} \cong R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q} .$$

Hence $H^{r-1}_{\mathfrak{q}R_\mathfrak{q}}(A_\mathfrak{q}) \cong R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q}$. This shows that $A_\mathfrak{q}$ is Buchsbaum and not Cohen–Macaulay.

Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \not\supseteq V(\mathfrak{q}) := \operatorname{Supp}(A/\mathfrak{q})$. We get again from (9) that $H^{r-\dim A/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(A_\mathfrak{p}) = 0$, that is, $A_\mathfrak{p}$ is always Cohen–Macaulay.

Assume now that A is locally Buchsbaum but not locally Cohen–Macaulay. Hence A is not Buchsbaum. Our Theorem B(ii) shows that $r = \operatorname{depth} A \geq 2$. Suppose that $r \geq 3$. Then we get from (9) $H^1_{\mathfrak{q}R_\mathfrak{q}}(A_\mathfrak{q}) \neq 0$. Since $\dim A/\mathfrak{q} = r - 1 \geq 2$ and $\dim A_\mathfrak{q} = d - (r-1) \geq 2$ we get that $A_\mathfrak{q}$ is not Cohen–Macaulay, that is, $A_\mathfrak{w}$ is even not locally Cohen–Macaulay for all prime ideals $\mathfrak{w} \supset \mathfrak{q}$ and $\mathfrak{w} \neq \mathfrak{q}$. Hence $A_\mathfrak{w}$ is not Buchsbaum for these primes $\mathfrak{w}$, and this is a contradiction. Hence we have $r = 2$. This completes the proof of (iii).

(iv) This result is well known; see, e.g., [26, Corollary 16]. This completes the proof of Theorem B. □

Remark. It follows from the proof of Theorem B(i) that there is a linear subspace L of $\mathbb{P}^n$ of dimension $(r-2)$ such that every subscheme under consideration is locally Buchsbaum but not locally Cohen–Macaulay at L and locally Cohen–Macaulay outside of L (see our Example 9 in Section 7).

6. Proof of Theorem A

We first consider the case $d = 1$. The implication (i)$\Rightarrow$(ii) follows from Theorem C(i) since (ii) consists only of case (a). The implication (ii)$\Rightarrow$(i) follows from the following observation:

Consider the exact sequence

$$0 \to A \to H^0(A) \to H^1_P(A) \to 0\,.$$

Then we have

$$\deg X = h_0(A) = \operatorname{rank}_k[H^0(A)]_1 = \operatorname{rank}_k[A]_1 + \operatorname{rank}_k[H^1_P(A)]_1$$

$$= \operatorname{rank}_k[A]_1 + 1 = 2 + \operatorname{codim} A = 2 + \operatorname{codim} X\,.$$

We assume that $d \geq 2$. First we suppose that (i) holds. Then $\operatorname{reg} A = 2$ by Theorem 3(i) and $H^i_P(A) = 0$ for all $i \neq r, d$ by Lemma 6(i). If A is Cohen–Macaulay, then (ii) follows from Lemma 6(iii). If $d \geq 4$ and $2 \leq r \leq d-2$, then (ii) follows from Theorem B(i) and Lemma 6(ii). If $d \geq 3$ and $r = d-1$, then we obtain (ii) from Lemma 6(ii) and the following calculation by applying Theorem 3(ii):

$$
\begin{aligned}
&\operatorname{rank}_k[H_P^{d-1}(A)]_{2-d} \\
&\quad = \operatorname{rank}_k[H^{d-2}(A)]_{2-d} - \operatorname{rank}_k[H_P^d(A)]_{2-d} \\
&\quad = (-1)^{d-2}((-1)^{d-2} \operatorname{rank}_k[H^{d-2}(A)]_{2-d} \\
&\qquad + (-1)^{d-1} \operatorname{rank}_k[H_P^{d-1}(A)]_{2-d}) \\
&\quad = (-1)^{d-2} h(2-d, A) \quad (\text{note that } [H^0(A)]_{2-d} = 0 \text{ since } d \geq 3) \\
&\quad = (-1)^{d-2}\left(-\binom{2-d+d-1-3}{d-1-2}\right) = (-1)^{d-2}\left(-\binom{-2}{d-3}\right) = d-2\,.
\end{aligned}
$$

If $d \geq 2$ and $r = 1$, then we get (ii) from Lemma 6(ii) since $[H_P^1(A)]_p = 0$ for all $p \leq 0$ by Lemma 1(ii). This shows (ii).

We now assume (ii). First we consider the case (c) but in assuming $d \geq 4$. Then we get for the Hilbert polynomial $h(p, A)$

$$
\begin{aligned}
h(p, A) = \operatorname{rank}_k[A]_p &+ (-1)^{d-2} \operatorname{rank}_k[H_P^{d-1}(A)]_p \\
&+ (-1)^{d-1} \operatorname{rank}[H_P^d(A)]_p\,.
\end{aligned}
$$

Hence we obtain

$$h(2-d, A) = (-1)^{d-2}(d-2)\,, \tag{10}$$

$$h(3-d, A) = (-1)^{d-2} \quad (\text{since } d \geq 4)\,, \tag{11}$$

$$h(p, A) = 0 \qquad \text{for } 4-d \leq p \leq -1\,, \tag{12}$$

$$h(0, A) = 1\,, \tag{13}$$

$$h(1, A) = n+1\,. \tag{14}$$

Using (12) we can write

$$h(p, A) = (p+1)\cdots(p+d-4)(\alpha p^3 + \beta p^2 + \gamma p + \delta)$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Q}$. The remaining conditions (13), (14), (11), (10) yield

$$
\begin{aligned}
&(d-4)!\delta = 1\,, \\
&(d-3)!(\alpha + \beta + \gamma + \delta) = n+1\,, \\
&(d-4)!(\alpha(3-d)^3 + \beta(3-d)^2 + \gamma(3-d) + \delta) = 1\,, \\
&(d-3)!(\alpha(2-d)^3 + \beta(2-d)^2 + \gamma(2-d) + \delta) = d-2\,.
\end{aligned}
$$

Solving this linear system of equations we get

$$(d-1)!\alpha = n+3-d\,,$$

that is, $h_0(A) = n+3-d$. Hence we have degree $X = h_0(A) = 2 + \operatorname{codim} A = 2 + \operatorname{codim} X$. This shows (i).

If $d = 3$ in (c), a similar calculation shows

$$h(-1, A) = -1\,, \qquad h(0, A) = 0\,, \qquad h(1, A) = n+1\,.$$

Whence $h(p, A) = \alpha p^2 + \beta p$ with $2\alpha = n = n+3-d$, and (i) holds again.

Next we consider the case (d). If $r = 2$, we get

$$\begin{aligned} h(p, A) &= \operatorname{rank}_k[A]_p - \operatorname{rank}_k[H_P^2(A)]_p + (-1)^{d-1}\operatorname{rank}_k[H_P^d(A)]_p \\ &= -1 \quad \text{for } 2-d \le p \le -1\,, \end{aligned}$$

and $h(0, A) = 0$, $h(1, A) = n+1$. Therefore, we can set

$$h(p, A) + 1 = (p+1)(p+2)\cdots(p+d-2)(\alpha p + \beta)\,, \quad \alpha, \beta \in \mathbb{Q}\,,$$

and we obtain

$$1 = (d-2)!\beta\,, \qquad n+1 = (d-1)!(\alpha+\beta)\,,$$

whence $(d-1)!\alpha = n+3-d$.

If $r > 2$ in (d), we set

$$F(p) := h(p, A) + (-1)^r \binom{-p}{r-2}\,.$$

Note that

$$(-1)^r \binom{-p}{r-2} = p(p+1)\cdots(p+r-3)/(r-2)!$$

is zero for $3-r \le p \le 0$. Therefore $F(p) = 0$ for $2-d \le p \le -1$, $F(0) = 1$, $F(1) = n+2$. It follows that

$$F(p) = (p+1)(p+2)\cdots(p+d-2)(\alpha p + \beta)$$

with $(d-1)!\alpha = n+3-d$. Clearly, α is also the leading coefficient of the Hilbert polynomial $h(p, A)$.

The cases (a) and (b) can be proved by the same method, and are easier. Thus (ii) implies (i), and this completes our proof of Theorem A. $\square$

7. Examples and problems

We discuss in conclusion some examples and open questions. Let k be an algebraically closed field.

Example 8. Take the interesting example of Stanley [19, 4.5]. Let $I := (x_0x_3, x_1x_3, x_0x_1x_2)$ be the ideal of the polynomial ring $R := k[x_0, x_1, x_2, x_3]$. The graded k-algebra $A := R/I$ is reduced and pure-dimensional such that $4 = h_0(A) = \operatorname{codim} A + 2$. Then A is a Cohen–Macaulay k-algebra, but A is not Gorenstein. Indeed, the subscheme X of $\mathbb{P}_k^3$ defined by I is not irreducible. Hence our Theorem B(iv) is not true in general without the assumption that X is irreducible. We note that in this example Proj A is even connected in codimension 1.

In connection with our remark after the proof of Theorem B we want to discuss the following example:

Example 9 (see also [21, p. 351] and [20, Example V.5.2]). Let $F \subset \mathbb{P}_k^4$ be the surface given parametrically by

$$(s^3, s^2t, stu, su(u-s)), u^2(u-s)) .$$

We know that degree $(F) = \operatorname{codim}(F) + 2 = 4$, and Sing $F = \{(1,0,0,0,0)\}$. Let $I(F)$ be the defining prime ideal of F in $R := k[x_0, \ldots, x_4]$. Set $A := R/I(F)$. Then dim $A = 3$ and depth $A = 2$. Let $\mathcal{O}$ be the local ring of F at the origin of $\mathbb{P}_k^4$. It therefore follows from Theorem B(iii) that $\mathcal{O}$ is not Cohen–Macaulay but a Buchsbaum ring. It follows from the remark after the proof of Theorem B that F has precisely one singularity which is Buchsbaum but not Cohen–Macaulay.

Nowadays it is easy to construct examples for the statement of Theorem B(ii) and (iii) by using, e.g., [24, 25, 11]. Here is an example for Theorem B(ii).

Example 10. Following [11] we consider the following surface F of $\mathbb{P}_k^5$ given parametrically by $(uv^3, uv^2w, v^4, v^3w, vw^3, w^4)$. Let $I(F)$ be the defining prime ideal of F in $R := k[x_0, \ldots, x_5]$. Then a run of the computer program MACAULAY [1] produced the minimal free resolution of $I(F)$. For instance, we have the following defining equation of F:

$$I(F) = (x_1x_2 - x_0x_3, x_1x_4 - x_0x_5, x_3x_4 - x_2x_5, x_3^3 - x_2^2x_4, x_1x_3^2 - x_0x_2x_4 ,$$

$$x_1^2x_3 - x_0^2x_4, x_2x_4^2 - x_3^2x_5, x_0x_4^2 - x_1x_3x_5, x_4^3 - x_3x_5^2) .$$

Set $A := R/I(F)$. It follows from [11] that $h_0(A) = 5 = \operatorname{codim} A + 2$, and depth $A = 1$. Hence our Theorem B(ii) shows that F is not arithmetically Cohen–Macaulay but Buchsbaum.

Finally, we want to pose the following two problems:

Problem 1. Describe the minimal free resolution of a reduced, irreducible and nondegenerate subvariety X of $\mathbb{P}_k^n$ with $\dim X \geq 1$ and $\mathrm{degree}(X) = \mathrm{codim}\, X + 2$.

We believe that the following lemma is useful in solving this problem.

Lemma 11. *Let $A := R/I$ be a graded k-algebra where I is a reduced and pure-dimensional homogeneous ideal of $R := k[x_0, \ldots, x_n]$. We assume that $d := \dim A \geq 2$, depth $A = 1$, $\mathrm{Proj}\, A$ is connected in codimension 1, $h_0(A) = \mathrm{codim}\, A + 2$, and $n = \mathrm{rank}_k[A]_1 - 1$.*

Then $\mathrm{Ext}_R^{n+1-d}(A, R)$ is a graded Cohen–Macaulay A-module with Krull-dim $\mathrm{Ext}_R^{n+1-d}(A, R) = d$ *generated by $[\mathrm{Ext}_R^{n+1-d}(A, R)]_{-n-2+d} \cong k^{n+2-d}$. Moreover, $\mathrm{Ext}_R^{n+1-d}(A, R)$ has a minimal free resolution of the following kind*:

$$
\begin{aligned}
0 &\to R \oplus R(1) \to R^{\mu_{n-d}}(2) \to \cdots \to R^{\mu_1}(n+1-d) \\
&\to R^{n+2-d}(n+2-d) \to \mathrm{Ext}_R^{n+1-d}(A, R) \to 0 .
\end{aligned}
$$

Proof. Lemma 6 shows that $H_P^1(A) \cong k(-1)$. Hence we may consider the exact sequence

$$0 \to A \to H^0(A) \to k(-1) \to 0 .$$

Since $\mathrm{Ext}_R^i(k(-1), R) = 0$ for all $i \neq n+1$ we get from this sequence

$$\mathrm{Ext}_R^{n+1-d}(A, R) \cong \mathrm{Ext}_R^{n+1-d}(H^0(A), R) .$$

The proof of Lemma 6 shows that $H^0(A)$ is a Cohen–Macaulay module over A of dimension d with

$$\mathrm{reg}\, H^0(A) = d + e(H_P^d(H^0(A))) = d + e(H_P^d(A)) \leq 1 .$$

Therefore, we get that $\mathrm{Ext}_R^{n+1-d}(A, R)$ also is a graded Cohen–Macaulay A-module. Moreover, similar methods as in our proof of Lemma 6 give the following minimal free resolution of $H^0(A)$:

$$0 \to R^{\lambda_{n+1-d}}(-n+d-2) \to \cdots \to R^{\lambda_1}(-2) \to R \oplus R(-1) \to H^0(A) \to 0 .$$

Hence by taking duals we get a minimal free resolution for $\mathrm{Ext}_R^{n+1-d}(H^0(A), R)$, and therefore also for $\mathrm{Ext}_R^{n+1-d}(A, R)$ as follows:

$$0 \to R \oplus R(1) \to R^{\mu_{n-d}}(2) \to \cdots$$
$$\to R^{\mu_0}(n+2-d) \to \operatorname{Ext}_R^{n+1-d}(A, R) \to 0\,,$$

where $\mu_i := \lambda_{n+1-d-i}$ for all $i = 0, \ldots, n-d$. This shows that $\operatorname{Ext}_R^{n+1-d}(A, R)$ is generated by

$$[\operatorname{Ext}_R^{n+1-d}(A, R)]_{-n+d-2} \cong [H_P^d(A)^\vee]_{d-1}$$
$$\cong \operatorname{Hom}_k([H_P^d(A)]_{1-d}, k) \cong k^{n+2-d}\,.$$

From this we also get $\mu_0 = \lambda_{n-d+1} = n + 2 - d$.

This shows Lemma 11. $\square$

Problem 2. Let X be a reduced, irreducible and nondegenerate subvariety of $\mathbb{P}_k^n$. Characterize all rational subvarieties X, that is, projective varieties birationally equivalent to $\mathbb{P}^m$ for some m, such that degree $X = \operatorname{codim} X + 2$.

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